Sequential Auctions and Auction Revenue

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Abstract. We consider the problem of a seller who owns K identical objects and N bidders each willing to buy at most one unit. The seller may auction the objects at two different dates. Assuming that buyer valuations are uniform and independent across periods, we show that the seller is better off by auctioning a positive number of objects in each period. We also provide sufficient conditions such that most objects should be auctioned at the first date or in the second date.

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1. Introduction

Consider a seller who owns K identical objects and must decide how to auction them to a group of N bidders willing to buy at most one unit. Suppose that the seller has a finite number of opportunities (two in our model) to sell the objects. We compare two different selling strategies: auction all N objects at once to K bidders; and auction first X objects to K bidders, then $N - X$ to $K - X$ bidders.[1] Real-world situations where this problem is relevant include auctions for treasury bills, radio spectrum, dairy products, and electricity supply.[2]

If buyer valuations remain constant across periods (or if they change but the relative ranking remains unchanged), then the classic Martingale theorem suggests that the seller is indifferent in how it divides the lots across auctions and periods. By contrast, we show that, if buyer valuations change over time, then the seller is better off by splitting its total assets into two different lots (i.e., $0 < X^* < K$): even if expected buyer *valuations* are the same in every period, expected *selling price* is higher with sequential auctions. Intuitively, what the seller cares for is not average valuation but rather the average value of the K highest valuations; and this order statistic is increasing in the number of valuation draws. In other words, by increasing the number of auction dates, the seller enlarges market size.

We next consider the optimal split of a seller's assets across two auctions. There are two effects at work here: a *market size* effect and a *strategic bidding* effect. Consider first the market size effect: since there are fewer bidders in the second auction (the winners of the first auction drop out), the seller (a monopolist with limited total capacity) will want to maximize a quantity-weighted average of the two auction prices, which would optimally bias the split towards the first auction. Consider now the strategic bidding effect: anticipating the option of waiting for the second auction, whether buyers should shade their bids during the first auction depends on expected prices in the two auctions which will also bias the seller toward the first auction (except with K/N is large). Accordingly, the seller should optimally bias the split towards the first auction so as to mitigate bidder incentives toward waiting for the second auction.

If the number of objects is very small with respect to the number of bidders, then bid shading is small, the market-size effect dominates, and $X^* > K/2$, that is, the seller is better off by selling most units during the first auction. By contrast, as $K \to N$, the option value of waiting is very high and bidding in the first auction non-aggressive. Accordingly, $X^* > K/2$, that is, the seller is better off by selling most units during the second auction.

This paper builds off several strands of literature: sequential auctions (e.g., Weber, 1983; Ashenfelter, 1989; De Silva et al., 2005; Donald et al., 2006; Hörner and Samuelson, 2011; Said, 2011, 2012); multi-unit auctions (e.g., McAdams, 2006, Back and Zender, 2001, LiCalzi and Pavan, 2005, Lengwiler, 1999); dynamic mechanism design (Garrett, 2017; Ely et al., 2017). Despite a variety of similarities, we are the first — to the best of our knowledge — to consider the problem of sequential auctions with changing buyer valuations.

1. We assume the seller must sell all K objects by the end of the second date.
2. See www.globaldairytrade.info/en/; Ito and Reguant (2014); and Joskow (1997).

2. Model and results

Consider a seller who owns K identical objects for which it has no value; and suppose there are $N > K$ bidders interested in at most one of the objects. Consistent with observed practice, we assume all auctions are run as generalized Vickrey auctions: whenever n objects are auctioned to $m > n$ bidders, the n highest bids are allocated the objects and pay a price equal to the $n+1$st bid.[3]

There are two periods, $t = 1$ and $t = 2$. In each period, Nature generates bidder valuations v_i, $i = 1, ..., N$, conditional on the bidder not having yet obtained an object. Specifically,

Assumption 1. *First- and second-period values, v_{i1} and v_{i2}, are independent and uniformly distributed: $v_{it} \sim U[0,1]$, $t = 1, 2$.*

(If a bidder purchases at $t = 1$, then its valuation at $t = 2$ is equal to zero.)[4]

The seller must decide how to distribute the K objects along two auction periods. Let X be the number of objects auctioned at $t = 1$. We assume the product being auctioned is perishable, or has limited shelf life, and so if the seller decides to auction X objects at $t = 1$ (where $X \in \{1, ..., K\}$), then the seller must auction $K - X$ objects at $t = 2$.[5] Total expected revenue from auctioning K objects over two auctions is given by

$$R(X) = X\,\mathbb{E}(p_1) + (K - X)\,\mathbb{E}(p_2)$$

where p_t is equilibrium price at time t and $\mathbb{E}$ is the expected value operator. We begin with a result that characterizes the expected revenue function $R(X)$:

Lemma 1. *Under Assumption 1, the seller's expected revenue is given by*

$$R(X) = \max\left\{0, X\left(\frac{N-X-1}{N+1} - \frac{(K-X)(K-X+1)}{2(N-X)(N-X+1)}\right)\right\} + (K-X)\,\frac{N-K-1}{N-X+1} \quad (1)$$

where each of the two terms on the RHS represent expected revenue in the first and in the second period, respectively.

Next we show that selling all K units in one auction results in (strictly) lower expected revenues than selling at least one unit in each auction. We will consider the case when the value of N is large and make the following assumption:

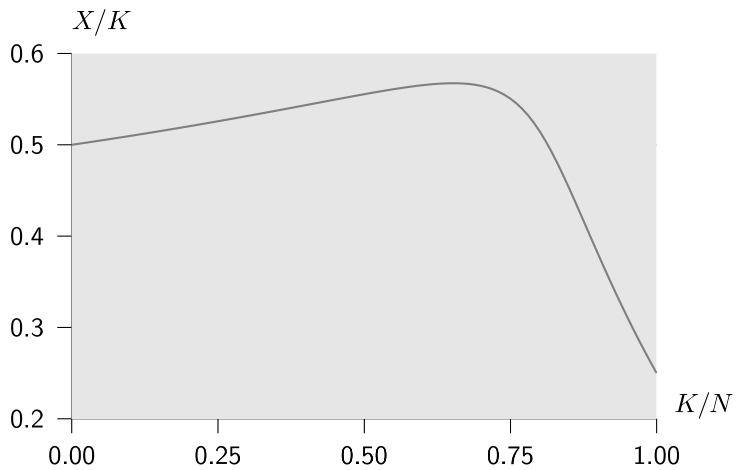
Assumption 2. $0 < \lim\limits_{N\to\infty} \dfrac{K}{N} < 1$

Let X^* be the revenue-maximizing value of X. Our first main result is that it is optimal for the seller to use both auction periods.

Proposition 1. *There exists $N' > 0$ such that, if $N > N'$, then $0 < X^* < K$*

3. We assume a zero reserve price.
4. Generally, we would expect a positive but less than perfect correlation; our choice of independence is essentially justified by expediency.
5. This is the case in both energy procurement and of auctions of perishable but storable commodities (e.g., Fonterra's global dairy trade (GDT) auctions).

Figure 1
Optimal X/K when $N \to \infty$



Having established that it is optimal to sell some objects at $t = 1$ *and* at $t = 2$, we next address the question of the relation between X and $K/2$: should the seller offer most objects at $t = 1$ or at $t = 2$?[6]

Proposition 2. *There exist $N' > 0$ and $0 < \alpha' < \alpha'' < 1$ such that, if $N > N'$ and $K/N < \alpha'$ (resp. $K/N > \alpha''$), then $X^* > K/2$ (resp. $X^* < K/2$).*

Figure 1 illustrates Propositions 1 and 2. It depicts the optimal X/K as a function of K/N when $N \to \infty$. First, notice that $X/K \in (0, 1)$ for all values of $K/N \in [0, 1]$ (Proposition 1). Second, $X/K > \frac{1}{2}$ — that is, it is optimal to sell most of the objects at $t = 1$ — if and only if K/N is not too large. The intuition is that there are two opposite effects when it comes to the optimal value of X: a *market size* effect and a *strategic bidding* effect. Suppose first that bidders are myopic, that is, do not take into account at $t = 1$ the option value of waiting until $t = 2$. Then the difference between auctions is that the second has a lower number of potential bidders. This market size effect implies that, at the margin, average price is higher when more objects are sold, that is, at $t = 1$. This is analogous to a monopolist with limited capacity selling in two markets with linear demands that have the same vertical-axis intercept and differ only in their horizontal-axis intercept: the optimal division of capacity is to sell more in the market with higher demand.

However, consumers are not myopic: when they bid at $t = 1$ they take into account the option value of waiting until $t = 2$. This option value leads buyers to shade their bids at $t = 1$, as shown earlier. Continuing with the example of a monopolist with limited capacity, this is equivalent to a downward shift in the demand curve in one of the markets. If the number of bidders were the same in both periods, then this would imply a lower optimal price (and output level) in the first market.

If the number of objects is very small with respect to the number of bidders (low K/N), then the market size effect dominates. By contrast, if the number of objects is very large with respect to the number of bidders (high K/N), then the strategic bidding effect dominates.

6. If $K/N < 2$, then we have a more direct proof of Proposition 1. We also have a version of Proposition 1 which dispenses with the large N assumption but imposes limits on K/N.

Appendix

Proof of Lemma 1: Let X be the number of objects put up for auction (and sold) at $t = 1$. At $t = 2$, $N - X$ bidders compete for $Y = K - X$ units. The $K - X$ highest-value bidders win this auction and pay a price equal to the $(K - X + 1)$th highest bid. Since bidders bid their valuation, this corresponds to the $(N - X) - (K - X) = (N - K)$th order statistic of the set of bidder valuations. From Sections 25.3 and 26.4 of Johnson et al. (1995), the the expected value of this statistic, which is therefore the expected value of the second-auction price, is given by

$$\mathbb{E}(p_2) = \frac{N - K - 1}{N - X + 1} \tag{2}$$

Suppose a bidder has the ith highest bid in the second auction; and suppose that it is a winning bid, that is, $i \leq K - X$. The expected surplus from winning the auction as the ith bidder is given by the expected value of the difference between the $((N - X) - (i - 1))$th and the $(N - K)$th order order statistic. From Sections 25.3 and 26.4 of Johnson et al. (1995), this is given by

$$\mathbb{E}\big(v_i - v_{K-X+1}\big) = \frac{K - X + 1 - i}{N - X + 1}$$

The average value of i, conditional on being a winning bidder, is given by $((K - X) + 1)/2$. Substituting into the above equation, and multiplying by $(K - X)/(N - X)$ (the probability of being a winning bidder in the second period), we get the expected surplus from participating in the second auction. It is given by

$$\mathbb{E}(s_2) = \frac{(K - X)\,(K - X + 1)}{2\,(N - X)\,(N - X + 1)} \tag{3}$$

In the first-period auction, were it not for the opportunity value of bidding in the second auction, the X highest bids (the winning bids) would pay a price equal to the $(X + 1)$th highest bid, that is, the $N - X$ order statistic of bids (or valuations). Given the opportunity of bidding in the second auction, all of these bids are shaded by $\mathbb{E}(s_2)$. This implies an expected first-period price of

$$\mathbb{E}(p_1) = \frac{N - X - 1}{N + 1} - \mathbb{E}(s_2) \tag{4}$$

where $\mathbb{E}(s_2)$ is given by (3). The seller's problem is to maximize overall expected revenue, which is given by

$$R(X) = X\,\mathbb{E}(p_1) + (K - X)\,\mathbb{E}(p_2)$$

where $\mathbb{E}(p_1)$ is given by (4) and $\mathbb{E}(p_2)$ is given by (2). Finally, we must require that auction price in the first period be positive, which explains the min function in the result. ■

Proof of Proposition 1: Suppose that seller revenue during the first period is strictly positive (an assumption we will confirm later). Then from Lemma 1,

$$R(X) = X\left(\frac{N - X - 1}{N + 1} - \frac{(K - X)\,(K - X + 1)}{2\,(N - X)\,(N - X + 1)}\right) + (K - X)\,\frac{N - K - 1}{N - X + 1}$$

It follows that (since $N > 0$)

$$
\begin{aligned}
R(X)/N &= (X/N)\left(\frac{1 - X/N - 1/N}{1 + 1/N} - \frac{(K/N - X/N)\,(K/N - X/N + 1/N)}{2\,(1 - X/N)\,(1 - X/N + 1/N)}\right) \\
&\quad + (K/N - X/N)\,\frac{1 - K/N - 1/N}{1 - X/N + 1/N} \qquad (5)
\end{aligned}
$$

Since N is exogenously given, maximizing $R(X)$ or maximizing $R(X)/N$ yields the same result. Define

$$
\begin{aligned}
a &\equiv X/K \\
b &\equiv K/N
\end{aligned} \qquad (6)
$$

In words, b is a parameter indicating the ratio of the number of objects to the number of bidders, where we note that, by Assumption 2, in the limit as $N \to \infty$, $b \in (0, 1)$. The value of a, in turn, is a sufficient statistic of the choice variable X: it indicates the fraction of objects to sell at $t = 1$. Our goal is therefore to establish that $a \in (0, 1)$, that is, the seller is better off by using the two sell dates. Before continuing, note that

$$
X/N = a\,b \qquad (7)
$$

Substituting (6) and (7) into (5)

$$
R(X)/N = a\,b\left(\frac{1 - a\,b - 1/N}{1 + 1/N} - \frac{(b - a\,b)\,(b - a\,b + 1/N)}{2\,(1 - a\,b)\,(1 - a\,b + 1/N)}\right) + (b - a\,b)\,\frac{1 - b - 1/N}{1 - a\,b + 1/N}
$$

Define

$$
\begin{aligned}
r(a, b) &\equiv \lim_{N \to \infty} R(X)/N \\
&= a\,b\left(\frac{1 - a\,b}{1} - \frac{(b - a\,b)\,(b - a\,b)}{2\,(1 - a\,b)\,(1 - a\,b)}\right) + (b - a\,b)\,\frac{1 - b}{1 - a\,b} \\
&= a\,b\,(1 - a\,b) - \frac{a\,b^3\,(1 - a)^2}{2\,(1 - a\,b)^2} + b\,(1 - a)\,(1 - b)/(1 - a\,b) \qquad (8)
\end{aligned}
$$

We now show that the value of a that maximizes $r(a, b)$ is strictly between 0 and 1. Since $r(a, b)$ results from the sum, multiplication and division of polynomials and denominators are different from zero (because $b \in (0, 1)$), we conclude that $r(a, b)$ is continuous and differentiable. Computation establishes that

$$
dr/da\,\big|_{\,a\,=\,0} = b^2\left(2 - \tfrac{3}{2}\,b\right) > 0
$$

where the inequality follows from $b \in (0, 1)$. Given continuity and Taylor's theorem, we conclude that, in the neighborhood of $a = 0$, $r(a, b)$ is strictly increasing in a. This implies that $R(1) > R(0)$. Moreover, since $R(K) = R(0)$ we conclude that $R(1) > R(K)$ as well.

Finally, we need to show that, as assumed above, the constraint

$$
X\left(\frac{N - X - 1}{N + 1} - \frac{(K - X)\,(K - X + 1)}{2\,(N - X)\,(N - X + 1)}\right) > 0
$$

is not binding; or, equivalently (in the $N \to \infty$ limit),

$$a\,b\,(1 - a\,b) - \frac{a\,b^3\,(1-a)^2}{2\,(1-a\,b)^2} > 0 \tag{9}$$

At $a = 0$, the left-hand side of (9) equals zero. Moreover, the derivative of the left-hand side with respect to a, evaluated at $a = 0$, is equal to $b\left(1 - b^2/2\right)$, which is strictly positive since $b \in (0,1)$. This confirms our assumption that first-period price is positive. ■

Proof of Proposition 2: Suppose that seller revenue during the first period is strictly positive (an assumption we will confirm later). Define

$$f_n(a) \equiv \left.\frac{d^n\left(dr/da\right)}{db^n}\right|_{b=0}$$

where r is defined by (8) and $f_0(a)$ is simply dr/da. Computation establishes that

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 0 \\ f_2 &= 4 - 8\,a \\ f_3 &= 36\,a - 27\,a^2 - 9 \end{aligned}$$

Continuity and Taylor's theorem imply that, in the neighborhood of $b = 0$, the optimal value of a lies in the neighborhood of $a = \frac{1}{2}$. Moreover, $f_3\left(\frac{1}{2}\right)\frac{9}{4} > 0$. Continuity and Taylor's theorem imply that, in the neighborhood of $b = 0$, the optimal value of a is strictly increasing in b. We still need to confirm that our assumption that the first-period price is positive is justified. Substituting $a = \frac{1}{2}$ into the left-hand side of (9) we get

$$\left.\left(a\,b\,(1 - a\,b) - \frac{a\,b^3\,(1-a)^2}{2\,(1-a\,b)^2}\right)\right|_{a=\frac{1}{2}} = \tfrac{1}{4}\,\frac{8 - 4\,b + b^2}{4 - 4\,b + b^2}\,b\,(1-b)$$

which is strictly positive for $b \in (0,1)$. It follows that it is strictly greater than $\frac{1}{2}$.

Further computation establishes that

$$\left.\frac{dr}{da}\right|_{b=1} = \tfrac{1}{2} - 2\,a$$

which implies $a = \frac{1}{4}$. We still need to confirm that our assumption that the first-period price is positive is justified. Substituting $a = \frac{1}{4}$ into the left-hand side of (9) we get

$$\left.\left(a\,b\,(1 - a\,b) - \frac{a\,b^3\,(1-a)^2}{2\,(1-a\,b)^2}\right)\right|_{a=\frac{1}{4}} = \frac{b}{\left(1 - \frac{1}{4}\,b\right)^2}\left(\frac{1}{4} - \frac{3}{16}\,b - \frac{3}{128}\,b^2 - \frac{3}{256}\,b^3\right)$$

The first term on the right-hand side is positive for all $b \in (0,1)$. The second term is positive for $b = 1$ and strictly decreasing in b for $b \in (0,1)$. It follows that, in the neighborhood of $a = \frac{1}{4}$, first period price is strictly positive for all $b \in (0,1)$, as assumed. Finally, the result follows by continuity. ■

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