Sequential Auctions and Auction Revenue

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Abstract. We consider the problem of a seller who owns K identical objects and N bidders willing to buy at most one unit. The seller may auction the objects at two different moments in time. Assuming that buyer valuations are uniform and independent across periods, we show that the seller is better off by auctioning a positive number of objects in each period. We also provide sufficient conditions such that most objects should be auctioned in the first period or in the second period.

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1. Introduction

Each year since 2002, the four New Jersey Electric Distribution Companies (EDCs) have procured electric supply to serve their Basic Generation Service (BGS) customers through statewide auctions. These auctions are always for future delivery, typically 6 months or so in advance. One important feature is that the purchaser/auctioneer can defer some purchases for some months. Moreover, the time difference between auctions is such that there is a significant probability that the bidders' valuations will change in the meantime.

Specifically, suppose that the auctioneer does split the auction into two different stages and consider the problem faced by a seller/bidder who possesses one unit of capacity. Since the cost of supplying power at a future date includes the opportunity cost of not supplying it to an alternative buyer (e.g., an out-of-state buyer or a large customer in New Jersey), the seller/bidder's value at $t = 2$ may be different from the value at $t = 1$. Notice that this is not simply a matter of information about one's valuation: it's the valuation itself that varies.

There are many other real-world situations when an auctioneer must decide how to schedule the purchase or sale of a number of perfectly substitutable or nearly-perfectly-substitutable goods; and where bidder valuations may vary over time. In addition to electricity supply, examples include treasury bills, radio spectrum and dairy products.[1]

Sometimes — as in the New Jersey electricity supply case — the auctioneer is a buyer; in other cases — as in government radio spectrum auctions — the auctioneer is a seller. The problem faced by the auctioneer is similar, and for simplicity sake we focus on the latter. We consider a seller who owns K identical objects and must decide how to auction them to a group of N bidders willing to buy at most one unit. Specifically, we compare two different selling strategies: auction all N objects at once to K bidders; and auction first X objects to K bidders, then $N - X$ to $K - X$ bidders.[2]

We show that, if buyer valuations remain unchanged from period to period, then it does not matter (in terms of expected price and revenues) how the auction is scheduled (that is, expected price is invariant with respect to X). However, if buyer valuations change over time, then the seller is better off by splitting its total assets into two different lots (i.e., $0 < X^* < K$): even if expected buyer *valuations* are the same in every period, expected *selling price* is higher with sequential auctions. Intuitively, what the seller cares for is not average valuation but rather the average value of the K highest valuations; and this extreme statistic is increasing in the number of valuation draws. In other words, by increasing the number of auction dates, the seller enlarges market size.

Having established that the seller is better off with sequential auctions, we move on to consider the problem of optimal splitting of the seller's assets. We provide sufficient conditions such that the seller is better off selling most objects in the first auction; and sufficient conditions such that the seller is better off selling most objects in the second auction. There are two effects at work here: a *market size* effect and a *strategic bidding* effect. Consider first the market size effect: since there are fewer bidders in the second auction (the winners of the first auction drop out), the seller (a monopolist with limited total capacity) should optimally bias the split towards the first auction. Consider now the strategic bidding effect: anticipating the option of waiting for the second auction, buyers

1. See www.globaldairytrade.info/en/; Ito and Reguant (2014); and Joskow (1997).

2. We assume the seller must sell all K objects by the end of the second date.

shade their bids during the first auction. Accordingly, the seller should optimally bias the split towards the second auction.

If the number of objects is very small with respect to the number of bidders, then bid shading is small, the market-size effect dominates, and $X^* > K/2$, that is, the seller is better off by selling most units during the first auction. By contrast, as $K \to N$, the option value of waiting is very high and bidding in the first auction non-aggressive. Accordingly, $X^* > K/2$, that is, seller is better off by selling most units during the second auction.

■ **Related literature.** This paper builds off at least three separate strands of literature that have studied both the theory of and experience with sequential auctions. On the theory end, Weber (1983) shows that in a sequence of auctions of identical objects in which bidders have independent private values, are risk neutral, and have valuations which do not change across auctions, the expected price in each auction is the realized price of the previous auction. That is, on average, there should be no upwards or downwards trend in prices; the expected price is a martingale. The intuition is that the marginal bidder in each auction, t, will bid based on what it expects price to be auction $t + 1$.[3]

In contrast with this theoretical prediction, the empirical literature has found a variety of price patterns. Ashenfelter (1989) observed that, in sequences of wine auctions, prices tend to decline over time more often than not.[4] Subsequently, similar results were shown in auctions of art, real estate, and cattle.[5] By contrast, Donald et al. (2006) examine sequential auctions of timber in Russia and find that a revenue equivalence results holds. More specifically, they find prices are a martingale when bidders each demand one unit, but is a submartingale when bidders have multi-unit demands.[6]

The empirical research, especially that which identifies a declining price "anomaly," has given rise to a large strand of research aimed at deriving the theoretical conditions under which prices will decrease, or increase, across a sequence of auctions. McAfee and Vincent (1993), for example, find that prices will decrease if bidders are risk averse: early bids are equal to expected later prices plus a risk premium.[7] Their explanation relies on the assumption of non-decreasing absolute risk aversion. Conversely, Milgrom and Weber (1999) suggest that prices will tend to *increase*, due to more aggressive bidding in late auctions.[8] Bernhardt and Scoones (1994) look at two sequential auctions with stochastically equivalent values. They find that even though bidders are risk-neutral, prices fall. Intuitively, bidders with higher valuations in the first auction discount their bids (due to the option value of participating in later auctions) less than those with lower valuations, and it is this first group that determines the price in the first auction, leading to lower prices in subsequent auctions.[9] Engelbrecht-Wiggans (1994) finds that prices will decline in a model similar to

3. See Milgrom and Weber (1982).
4. Ginsburgh (1998) attributes this price pattern to absentee bidders using proxies.
5. Ashenfelter and Graddy (2003), Lyk-Jensen and Chanel (2007), Beggs and Graddy (1997), and Pesando and Shum (2008) studied art and collectibles, Ashenfelter and Genesove (1992) studied real estate auctions, Engelbrecht-Wiggans and Kahn (1992) looked at cattle auctions, and De Silva et al. (2005) at construction contracts.
6. Also see Mezzetti (2011). Also, Lyk-Jensen and Chanel (2007) look at price trends when retailers, who have multi-unit demands, compete against consumers each bidding for only a single unit.
7. See also Buccola (1982).
8. See also Krishna (2009), Chapter 15.
9. See also Engelbrecht-Wiggans (1993)

this. Prices will also fall when supply is uncertain.[10]

A related line of research on multi-unit auctions, in which bids are demand functions, has found that bidder ability to benefit from strategic withholding in multi-unit auctions can be attenuated when the auction manager can adjust the auction volume.[11] For example, in a uniform-price auction, bidders compete by simultaneously submitting their demand schedules for the divisible good on offer. The seller compares aggregate demand with the auction supply and computes the clearing price, which is then paid by all bidders. Uniform-price auctions of a divisible good in fixed supply may lead to bidders strategically submitting high infra-marginal bids, resulting in a lower clearing price. When a bidder can be pivotal, the auction manager or auction designer can offset the strategic withholding incentives of such a bidder by making the auction supply or demand variable. The basic idea is that a bidder's market power is reduced when the supply or demand schedule it faces is more elastic. This assumes that any quantities not purchased or sold in the given auction will never be re-auctioned.

In contrast, this paper assumes each bidder can purchase only one unit and that the total quantity to be sold is fixed.[12] Perhaps the closest parallel to this paper is Hörner and Samuelson (2011), which addresses the problem of a monopolist with a fixed amount to sell to a pool of buyers and who must sell off its inventory by a specific date. They characterize optimal price paths where buyers have static values. They show that the optimal policy replicates a Dutch auction when there are many buyers. By contrast, we consider the case when buyer values evolve over time.

Finally, Said (2011) and Said (2012) also looks at sequential auctions with stochastically equivalent distributions of bidder values. However, in one case, Said assumes that the pool of bidders is the same size in each auction — that is bidders who win or leave one auction are replaced by new bidders in the next. In contrast, we assume a fixed pool of bidders.

2. Model

Consider a seller who owns K identical objects for which it has no value; and suppose there are $N > K$ bidders interested in at most one of the objects. Consistent with the observed practice, we assume all auctions are run as generalized Vickrey auctions: whenever n objects are auctioned to $m > n$ bidders, the n highest bids are allocated the objects and pay a price equal to the $n + 1$st bid.

There are two periods, $t = 1$ and $t = 2$. In each period, Nature generates bidder valuations v_i, $i = 1, ..., N$. (If a bidder purchases at $t = 1$, then its valuation at $t = 2$ is equal to zero.) For most of the paper, we consider the case when $t = 2$ valuations are identically and independently distributed with respect to $t = 1$ valuations; that is, v_{it}, bidder i's valuation at time t, is drawn from $F(v_{it})$.

10. For a general model see Kittsteiner et al. (2004). Related papers include Branco (1997), Hausch (1988) and Sørensen (2006). And Hausch (1988) looks at bid shading to deceive rivals in a model in which bidders may want multiple units. See also Katzman (1999), Gale and Stegeman (2001) and Mezzetti (2011). See also Jeitschko (1999).
11. McAdams (2007), Back and Zender (2001), LiCalzi and Pavan (2005), Lengwiler (1999), Engelbrecht-Wiggans and Kahn (1992).
12. Hörner and Samuelson (2011) look at pricing over time. Here, the amount for sale or purchase at any point in time is a variable, and the auctions occur at fixed discrete dates rather than continuously over time.

The seller must decide how to distribute the K objects along two auction periods. We assume the seller cannot commit to not selling some of the objects. In other words, if the seller decides to auction X objects at $t=1$ (where $X \in \{1, ..., K\}$), then the seller must auction $K-X$ objects at $t=2$.[13]

Finally, we assume bidders are fully rational, utility maximizing agents. Specifically, at $t=2$ bids are chosen so as to maximize the probability of having a winning bid times expected surplus (value minus bid) conditional on having a winning bid. As is well known this implies bidding one's valuation. At $t=1$, by contrast, rationality implies selecting a bid that maximizes total expected utility. This is given by the probability of having a winning bid times expected surplus (value minus bid) conditional on having a winning bid; plus the probability of not having a winning bid times the expected surplus from participating in the $t=2$ auction.

3. Example

In this section we provide a simple example that illustrates the idea that a seller is better off by spreading its sale over two periods rather than one. In the next section we generalize this result and also address the issue of the optimal split between two periods.

Suppose valuations are either 0 or 1 (the latter with probability α). Suppose there are two objects and three bidders willing to purchase at most one object. Recall that all auctions are generalized second-price auctions. If the three objects are auctioned at the same time, then price is equal to 1 only if all three valuations are high, which happens with probability α^3. It follows that expected price is given by $\overline{p} = \alpha^3$, whereas expected revenue (from selling two objects) is given by

$$R_2 = 2\,\alpha^3 \tag{1}$$

Now suppose one object is sold at $t=1$ and one at $t=2$. In the second period, there will be two bidders competing for one object. It follows that the expect price is given by $\overline{p} = \alpha^2$. Ex-ante (that is, before second period valuations are observed), a bidder expects a surplus only with its valuation is high (which happens with probability α) and price is low (which happens when the other bidder's valuation is low, which in turn happens with probability $1-\alpha$). It follows that the bidder's second auction expected surplus is $s_2 = \alpha\,(1-\alpha)$.

In the first period there are three bidders and one one object on sale — and the option of waiting until the second period. If a bidder's first-period valuation is high, then the optimal bid is $1 - s_2 = 1 - \alpha\,(1-\alpha)$; that is, the bidder bids its first-period valuation minus the expected surplus from waiting until the next auction. By contrast, if valuation is low, then the optimal bid is zero. Equilibrium price is given by the high bid if the number of high valuations is greater than one, which happens with probability $\alpha^3 + 3\,\alpha^2\,(1-\alpha) = \alpha^2\,(3 - 2\,\alpha)$; and zero otherwise. It follows that expected price in the first period is given by $\alpha^2\,(3-2\,\alpha)\,\big(1-\alpha\,(1-\alpha)\big)$. Finally, overall expected revenue from selling one unit at a time is

$$R_{1+1} = \alpha^2\,(3-2\,\alpha)\,\big(1-\alpha\,(1-\alpha)\big) + \alpha^2 \tag{2}$$

13. This is the case in both energy procurement and of auctions of perishable but storable commodities (e.g., Fonterra's global dairy trade (GDT) auctions).

What is the relation between R_2 and R_{1+1}? There are two effects to consider. First, controlling for buyer's inter-temporal strategic behavior, by restricting the number the number of objects the auctioneer elicits higher bids. This corresponds to the fact that $\alpha^2\,(3-2\,\alpha)$, from the first term in (1), is greater than α^3, from the first term of the expansion of (2) into $\alpha^3+\alpha^3$; and from the fact that α^2, from the second term in (1), is greater than α^3, from the second term of the expansion of (2) into $\alpha^3+\alpha^3$. Against this positive effect of scheduling two auctions we must account for bidder strategic behavior, namely the fact that bidders lower their first-auction bids in anticipation of the option value of bidding in the second auction. This corresponds to the second part of the first term in (2), $1-\alpha\,(1-\alpha)$, which is lower than 1.

Computation establishes that

$$R_{1+1}-R_2=\alpha^2\,(1-\alpha)\left(4-3\,\alpha+2\,\alpha^2\right)>\alpha^2\,(1-\alpha)>0$$

so expected revenue is higher under sequential auctions. In other words, the positive effect from restricting the number of objects in each period outweighs the negative effect from bidder strategic behavior.

Admittedly, this is a very simplistic example, with a highly stylized distribution of bidder values. In the next section we consider a more general setting and show that the same result applies: sequential auctions perform better than one simultaneous auction.

4. Optimal sequential auctions

In this section we provide a series of more general results regarding the optimality of multi-period auctioning. Suppose that a seller has K objects and two periods when to sell them. Is the seller better off selling all of the objects in one period, or rather selling X in the first period and $K-X$ in the second period?

As a reference point, we begin by arguing that, if valuations are constant, then a straightforward implications of the martingale theorem is that seller revenue is invariant (in expected value) with respect to how K objects are auctioned over the two periods. The argument runs as follows. In the second period, each remaining bidder bids its valuation. It follows that second period price is given by the $(K+1)$st highest valuation, a value that is independent of X. In the first period, would-be winning bidders have the option of waiting for the second period and pay the $(K+1)$st valuation (if they bid more that that valuation).

Specifically, first-period price is determined by the bid submitted by the $(X+1)$st highest valuation. This bidder should submit a bid equal to the expected value of the $(K+1)$st highest valuation, conditional on its valuation and conditional on being the $(X+1)$st highest valuation. In fact, by the same argument as in a simple second-price auction, submitting this bid is strictly better than submitting a higher bid, which would risk paying a higher price (in expected value). Submitting a lower bid leads to the same expected value, for by losing the first auction the $(X+1)$st bidder correctly expects, with probability 1, to be the highest bidder in the second period and winning an object at a price equal to the $(K+1)$st valuation.[14]

14. We are assuming that valuations belong to a dense subset of $\mathbb{R}$, so that the probability of a tie in valuations is zero. If the set of valuations is countable, then there is a strictly positive probability of a tie, but even then the bidder is indifferent (in expected value) between winning in at $t=1$ and winning at $t=2$.

Finally, the $t = 2$ expected price at $t = 1$ is given by the $(X + 1)$st highest valuation bidder's expected price, which in turn is equal to the actual price at $t = 1$. It follows that equilibrium price follows a martingale (for a formal proof, see Weber (1983)).[15]

In this section, we consider the opposite extreme assumption regarding valuations, namely that a given bidder's valuation in the second period is independent from the same bidder's valuation in the first period. Generally, we would expect a positive but less than perfect correlation; our choice of independence is essentially justified by expediency.

Assumption 1. *First- and second-period values, v_{i1} and v_{i2}, are independent and uniformly distributed: $v_{it} \sim U[0,1],\ t = 1,2$.*

Before getting to our main results, we present some useful lemmas pertaining to extreme statistics.

Lemma 1. *Suppose that $x \sim \mathbb{B}(a,b)$. Then*

$$\mathbb{E}(x) = \frac{a}{a+b}$$

where $\mathbb{B}$ is the Beta distribution and $\mathbb{E}$ the expected value operator.

The proof of this and the next results may be found in the appendix.

Lemma 2. *Let $x_1, ..., x_n$ be a random sample of random variables $x_i \sim \mathbb{U}(0,1)$. Let $y_1, ..., y_n$ be the order statistics of the sample, that is, $y_1 < y_2 < ... < y_n$. Then*

$$\begin{aligned} y_i &\sim \mathbb{B}(i, n-i+1) \\ y_i - y_j &\sim \mathbb{B}(i-j, n-i+j+1) \end{aligned}$$

where $\mathbb{U}$ stands for uniform and $\mathbb{B}$ for beta. Moreover,

$$\begin{aligned} \mathbb{E}(y_i) &= \frac{i}{n+1} \\ \mathbb{E}(y_i - y_j) &= \frac{i-j}{n+1} \end{aligned}$$

where $\mathbb{E}$ is the expected value operator.

With these results, we are now ready to attack the main question: what is an auctioneer's optimal distribution of objects across two auctions, one at $t = 1$ and one at $t = 2$. Specifically, let X be the number of objects auctioned at $t = 1$. Total expected revenue from auctioning K objects over two auctions is given by

$$R(X) = X\,\mathbb{E}(p_1) + (K - X)\,\mathbb{E}(p_2)$$

where p_t is equilibrium price at time t and $\mathbb{E}$ is the expected value operator. We begin with a result that characterizes the expected revenue function $R(X)$:

15. Our argument can be extended easily to the case when $t = 2$ valuations are equal to $t = 1$ valuations plus a fix shifter.

Proposition 1. *Under Assumption 1, the seller's expected revenue is given by*

$$R(X) = X\left(\frac{N-X}{N+1} - \frac{(K-X)\,(K-X+1)}{2\,(N-X)\,(N-X+1)}\right) + (K-X)\,\frac{N-K}{N-X+1}$$

Let X^* be the optimal value of X, that is, the revenue-maximizing number of objects to auction at $t=1$. The next two results characterize the optimal solution. First we show that $0 < X^* < K$. Next we provide conditions such that $X > K/2$ or $X < K/2$.

Proposition 2. *Under Assumption 1, selling all K units in one auction results in lower expected revenues than selling at least one unit in each auction.*

The intuition for Proposition 1 is best understood as we consider the case when $N \to \infty$. It can be shown that the expected surplus from participating in the second auction goes to zero at the rate of N^2. This is similar to the result that, under linear demand, consumer surplus decreases to zero at the rate of the square of price: there is both a price effect (higher) and a quantity effect (lower). Likewise, as the number of bidders increases, the chances that a given bidder is the winning bid declines linearly with N; and the price that a winning bid must pay increases linearly in N (approximately). So, while bidders are strategic, the relevance of bidder strategic behavior (shading bids at $t=1$) is of second order as $N \to \infty$.

Ignoring this effect (i.e., assuming that $\mathbb{E}(s_2) = 0$), we are left with two terms in the seller's revenue expression:

$$X\,\frac{N-X}{N+1}$$

(the seller's expected revenues at $t=1$) and

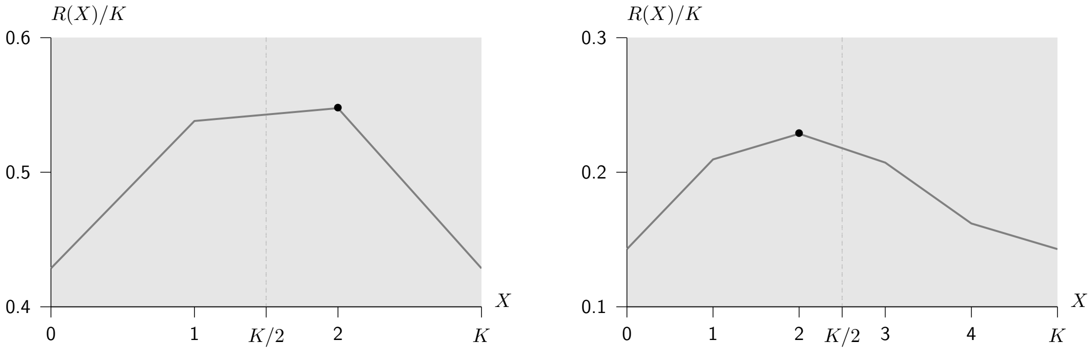
$$(K-X)\,\frac{N-K}{N-X+1}$$

(the seller's expected revenues at $t=2$). As a consequence, starting from $X=0$, an increase in X to $X=1$ implies an increase in first-period revenues that exceeds the decrease in $t=2$ revenues; and similarly, starting from $X=K$, a decrease in X to $X=K-1$ implies a decrease in first-periods revenues that falls short of the increase in $t=2$ revenues.

The argument is similar to a monopolist allocating limited fixed capacity across two markets. Assuming the demand curves in the two markets do not differ much, it is optimal to sell a strictly positive output in both markets. In fact, the two-market analogy also helps understand why a seller would benefit from running two auctions: the time variation in valuations effectively creates two different markets. (As shown at the beginning of the section, this is not true if valuations are perfectly correlated across periods.) To put it differently, the fact that twice as many valuations are generated increases the sellers expected revenue: although average valuations remain the same, the expected value of high order statistics (the ones that are relevant to an auctioneer) increase as we move from one to two auction dates.

Proposition 3. *Under Assumption 1, (a) $\exists\, K' \in \mathbb{N}$ such that, if $2 < K < K'$ and $N > 4$, then it is optimal to sell more units at $t=1$ than at $t=2$; (b) $\forall N > 4$, $\exists\, K'' \in \mathbb{N}$ such that, if $K > K''$, then it is optimal to sell more units at $t=2$ than at $t=1$.*

Figure 1
Expected revenue as a function of first auction size X, where $X \in \{0, ..., K\}$
Case (a) implies $X^* > \frac{1}{2}K$; Case (b) implies $X^* < \frac{1}{2}K$



(a) $N = 6,\ K = 3$ (a) $N = 6,\ K = 5$

The two panels in Figure 1 illustrate the two parts of Proposition 3. The left panel illustrates the case when K is small with respect to N, resulting in $X^* > \frac{1}{2}K$. The right panel illustrates the case when K is large with respect to N, resulting in $X^* < \frac{1}{2}K$

The intuition for Proposition 3 is that there are two opposite effects when it comes to the optimal value of X: a *market size* effect and a *strategic bidding* effect. Suppose first that bidders are myopic, that is, do not take into account at $t = 1$ the option value of waiting until $t = 2$. Then the difference between auctions is that the second has a lower number of potential bidders. This market size effect implies that, at the margin, it is better to put more objects for sale at $t = 1$ than at $t = 2$. This is analogous to a monopolist with limited capacity selling in two markets with linear demands that have the same vertical-axis intercept and differ only in their horizontal-axis intercept: the optimal division of capacity is to sell more in the market with higher demand.

However, consumers are not myopic: when they bid at $t = 1$ they take into account the option value of waiting until $t = 2$. This option value leads buyers to shade their bids at $t = 1$, as shown earlier. Continuing with the example of a monopolist with limited capacity, this is equivalent to a downward shift in the demand curve in one of the markets. If the number of bidders were the same in both periods, then this would imply a lower optimal price (and output level) in the first market.

If the number of objects is very small with respect to the number of bidders (case (a)), then the expected surplus from waiting until $t = 2$ is small; and the market size effect dominates. At the opposite extreme, as $K \to N$ (case (b)), the expected surplus in the $t = 2$ auction is $\frac{1}{2}$ (which is equal to $\mathbb{E}(v_2)$, as expected price is zero); and the strategic bidding effect dominates.

■ **Equilibrium prices.** Although our main focus is on expected revenue, our results also have implications for the equilibrium price path. Previous literature has addressed the empirical observation that sequential auctions for identical objects show a declining price path. When bidders' values do not change from one auction to the next, Weber's martingale Theorem applies — the expected price in one of a sequence of auctions is the realized price in the previous auction. This follows as the highest losing bidder in one auction will be the highest bidder in the next. Given this theoretical result, the frequently-observed declining price path is described as an "anomaly."

In our paper, bidders' relative values do change from one auction to the next. As a result, Weber's martingale Theorem fails to apply. This suggests a natural follow-up question: do prices tend to increase or decrease over time. The next limiting result provides an answer:

Proposition 4. *For each K, X such that $X < K/2$ (resp. $X > K/2$), there exits an N' such that, if $N > N'$, then expected price declines (resp. increases) from $t = 1$ to $t = 2$.*

In other words, Proposition 4 states that, if the seller decides to auction most objects at $t = 2$, then price at $t = 1$ is expected to be higher than at $t = 2$. Proposition 4 can be thought of as a companion to Proposition 3. The latter provides sufficient conditions such that it is optimal to auction a greater number of objects during the first (or during the second) of two auctions. If we assume that the seller behaves optimally (that is, selects the optimal X), then the normative result (Proposition 3) implies a positive result (Proposition 4).

■ **The n auction case.** In all of our previous results we considered the case when the seller holds at most two different auctions. We believe the qualitative nature of the results extends to the n-auction case. In what follows, we provide a result regarding the price path in the limit case when the seller holds a sequence of one-object auctions.[16]

Suppose that, in each of $K > 2$ periods, the seller auctions one object. Nature generates bidder valuations v_i, $i = 1, ..., N$, where $N > K$. We maintain Assumption 1, namely that valuations are uniformly distributed in [0,1], i.i.d. across periods. Moreover, each bidder has utility for one unit only: if a bidder at time t, then its valuation at $\tau > t$ is zero. Let $\mathbb{E}(p_t)$ be expected price of the t-th auction.

Proposition 5. $\mathbb{E}(p_{t+1}) \leq \mathbb{E}(p_t)$, *where* $1 \leq t < K$

The intuition for Proposition 5 is related to the market size effect mentioned earlier. As time goes by, the number of bidders decreases as successful bidders leave the auction. As a result, the second-order statistic of valuations also decreases, and so does equilibrium price.

5. Discussion and concluding remarks

This paper provides an analysis of a few selected types of sequential auctions. When bidder valuations all change in the same manner, at least in expectation, the martingale theorem applies, so that the expected price in one auction is the actual price in the most recent auction. Moreover, the auction manager's decision of how to split its available stock across auction dates does not affect total expected revenues. The main insight in this paper is that, if valuations change over time, the martingale theorem no longer holds. This motivates a normative theory of optimal scheduling of auctions and a positive theory of price paths. Specifically, we show that, if the number of objects is small with respect to the total number of bidders, than the seller is better off by selling most objects early on. We also provide sets of sufficient conditions such that prices decline over time (even though buyers are rational, forward-looking bidders).

16. A similar results appears in Engelbrecht-Wiggans (1994).

Appendix

Proof of Lemma 1: See Section 25.3 of Johnson et al. (1995). ■

Proof of Lemma 2: For the first part, see Section 26.4 of Johnson et al. (1995). The second part follows from Lemma 1. ■

Proof of Proposition 1: Let X be the number of objects put up for auction (and sold) at $t = 1$. At $t = 2$, $N - X$ bidders compete for $Y = K - X$ units. The $K - X$ highest-value bidders win this auction and pay a price equal to the $(K - X + 1)$th highest bid. Since bidders bid their valuation, this corresponds to the $((N - X) - (K - X)) = (N - K)$th order statistic of the set of bidder valuations. By Lemma 2, the the expected value of this statistic, which is therefore the expected value of the second-auction price, is given by

$$\mathbb{E}(p_2) = \frac{N - K}{N - X + 1} \tag{3}$$

Suppose a bidder has the ith highest bid in the second auction; and suppose that it is a winning bid, that is, $i \leq K - X$. The expected surplus from winning the auction as the ith bidder is given by the expected value of the difference between the $((N - X) - (i - 1))$th and the $(N - K)$th order order statistic. From Lemma 2, this is given by

$$\mathbb{E}\big(v_i - v_{K-X+1}\big) = \frac{K - X + 1 - i}{N - X + 1}$$

The average value of i, conditional on being a winning bidder, is given by $((K - X) + 1)/2$. Substituting into the above equation, and multiplying by $(K - X)/(N - X)$ (the probability of being a winning bidder in the second period), we get the expected surplus from participating in the second auction. It is given by

$$\mathbb{E}(s_2) = \frac{(K - X)\,(K - X + 1)}{2\,(N - X)\,(N - X + 1)} \tag{4}$$

In the first-period auction, were it not for the opportunity value of bidding in the second auction, the X highest bids (the winning bids) would pay a price equal to the $(X + 1)$th highest bid, that is, the $N - X$ order statistic of bids (or valuations). Given the opportunity of bidding in the second auction, all of these bids are shaded by $\mathbb{E}(s_2)$. This implies an expected first-period price of

$$\mathbb{E}(p_1) = \frac{N - X}{N + 1} - \mathbb{E}(s_2) \tag{5}$$

where $\mathbb{E}(s_2)$ is given by (4). The seller's problem is to maximize overall expected revenue, which is given by

$$R(X) = X\,\mathbb{E}(p_1) + (K - X)\,\mathbb{E}(p_2)$$

where $\mathbb{E}(p_1)$ is given by (5) and $\mathbb{E}(p_2)$ is given by (3). ■

Proof of Proposition 2: Define

$$\Delta \equiv R(1) - R(0)$$

Analytical computation establishes that

$$\Delta\,|_{\,K\,=\,2}\; > 0, \quad \Delta\,|_{\,K\,=\,N}\; > 0, \quad \left.\frac{d\Delta}{dK}\right|_{\,K\,=\,2} > 0, \quad \left.\frac{d\Delta}{dK}\right|_{\,K\,=\,N} < 0, \quad \frac{d^2\Delta}{dK^2} < 0$$

for all $N > 2$. Together, these imply that $\Delta > 0$, and the result follows. ■

Proof of Proposition 3: Computation establishes that

$$R(2) - R(1)\,|_{\,K\,=\,3} \;= \frac{3\,(N^2 - 5\,N + 2)}{N\,(N+1)\,(N-1)\,(N-2)}$$

which is positive for $N > 4$. This proves part (a). Computation also establishes that, if $K = N$, then $R(X)$ is a convex function of X with a unique maximum at $X = (N-1)/4 < N/2$. This proves part (b). ■

Proof of Proposition 4: From the proof of Proposition

$$\mathbb{E}(p_1) = \frac{N - X}{N + 1} - \frac{(K - X)\,(K - X + 1)}{2\,(N - X)\,(N - X + 1)}$$
$$\mathbb{E}(p_2) = \frac{N - K}{N - X + 1}$$

This implies that

$$\mathbb{E}(p_2) - \mathbb{E}(p_1) = A\,(B + C)$$

where $A > 0$ and $A = \mathbb{O}(n^{-3})$, $C = \mathbb{O}(n)$ and

$$B = 4\,n^2\,\big(K/2 - X\big)$$

where $\mathbb{O}$ is the order-of-magnitude operator. The result follows. ■

Proof of Proposition 5: Consider the auction for the last object. There are m remaining bidders, where $m \equiv N - K + 1$. Expected price is then given by

$$\mathbb{E}(p_{_K}) = \frac{m - 1}{m + 1}$$

Expected surplus for each the m bidders is

$$\mathbb{E}(s_{_K}) = \frac{1}{m\,(m+1)}$$

It follows that the expected value of the next to last auction price is

$$\mathbb{E}(p_{_{K-1}}) = \frac{m}{m + 2} - \frac{1}{m\,(m+1)}$$

The condition that $\mathbb{E}(p_{_{K-1}}) > \mathbb{E}(p_{_K})$ is equivalent to $3\,m^2 + m > 2$, which true for all $m \geq 1$.

More generally, consider auction, $k - h$, that is the h^{th} before the last, $h = 1, 2, \ldots, k-1$ auction. Notice that the price will be the value of the second highest value remaining bidder plus the surplus that bidder will earn in subsequent auctions should it lose. The high and next to highest value bidders valuations when m bidders remain are $\frac{m}{m+1}$ and $\frac{m-1}{m+1}$. So, the high value bidder should expect a surplus of $\frac{1}{m+1}$ plus the surplus the losers will expect to derive in the subsequent auctions. So, the price in auction $k-h$ will be at least as large as the price in the next auction $k-h+1$ whenever $\frac{m-1}{m+1} + (\frac{1}{m}) \times (\frac{1}{m-1}) \geq \frac{m-2}{m}$, as the surplus in subsequent auctions is the same. But this is always the case when there is competition in subsequent auctions, i.e., $k \geq 2$. ■

References

Ashenfelter, O. "How Auctions Work for Wine and Art." *Journal of Economic Perspectives*, Vol. 3 (1989), pp. 23–36.

Ashenfelter, O. and Genesove, D. "Testing for Price Anomalies in Real Estate Auctions." Nber working paper # 4036 (1992).

Ashenfelter, O. and Graddy, K. "Auctions and the Price of Art." *Journal of Economic Literature*, Vol. 41 (2003), pp. 763–787.

Back, K. and Zender, J.F. "Auctions of divisible goods with endogenous supply." *Economics Letters*, Vol. 73 (2001), pp. 29 – 34.

Beggs, A. and Graddy, K. "Declining Values and the Afternoon Effect: Evidence from Art Auctions." *The Rand Journal of Economics*, (1997), pp. 544–565.

Bernhardt, D. and Scoones, D. "A Note on Sequential Auctions." *American Economic Review*, Vol. 84 (1994), pp. 653–57.

Branco, F. "Sequential Auctions with Synergies: An Example." *Economics Letters*, Vol. 54 (1997), pp. 159–163.

Buccola, S.T. "Price Trends at Livestock Auctions." *American Journal of Agricultural Economics*, Vol. 64 (1982), pp. 63–69.

De Silva, D.G., Jeitschko, T.D., and Kosmopoulou, G. "Stochastic Synergies in Sequential Auctions." *International Journal of Industrial Organization*, Vol. 23 (2005), pp. 183–201.

Donald, S.G., Paarsch, H.J., and Robert, J. "An Empirical Model of the Multi-Unit, Sequential, Clock Auction." *Journal of Applied Econometrics*, Vol. 21 (2006), pp. 1221–1247.

Engelbrecht-Wiggans, R. "Optimal Auctions Revisited." *Games and Economic Behavior*, Vol. 5 (1993), pp. 227–239.

—. "Sequential Auctions of Stochastically Equivalent Objects." *Economics Letters*, Vol. 44 (1994), pp. 87–90.

Engelbrecht-Wiggans, R. and Kahn, C. "An Empirical Analysis of the Afternoon Effect in Dairy Cattle Auctions." University of illinois working paper (1992).

Gale, I.L. and Stegeman, M. "Sequential Auctions of Endogenously Valued Objects." *Games and Economic Behavior*, Vol. 36 (2001), pp. 74–103.

Ginsburgh, V. "Absentee Bidders and the Declining Price Anomaly in Wine Auctions ." *Journal of Political Economy*, Vol. 106 (1998), pp. 1302–1319.

Hausch, D.B. "A Model of Sequential Auctions." *Economics Letters*, Vol. 26 (1988), pp. 227–233.

Hörner, J. and Samuelson, L. “Managing Strategic Buyers.” *Journal of Political Economy*, Vol. 119 (2011), pp. 379–425.

Ito, K. and Reguant, M. “Sequential Markets, Market Power and Arbitrage.” NBER Working Papers 20782, National Bureau of Economic Research, Inc (2014).

Jeitschko, T.D. “Equilibrium Price Paths in Sequential Auctions with Stochastic Supply.” *Economics Letters*, Vol. 64 (1999), pp. 67–72.

Johnson, N.L., Kotz, S., and Balakrishnan, N. *Continuous Univariate Distributions*. New York: John Wiley & Sons, 1995.

Joskow, P.L. “Restructuring, Competition and Regulatory Reform in the U.S. Electricity Sector.” *Journal of Economic Perspectives*, Vol. 11 (1997), pp. 119–138.

Katzman, B. “A Two Stage Sequential Auction with Multi-Unit Demands.” *Journal of Economic Theory*, Vol. 86 (1999), pp. 77–99.

Kittsteiner, T., Nikutta, J., and Winter, E. “Declining Valuations in Sequential Auctions.” *International Journal of Game Theory*, Vol. 33 (2004), pp. 89–106.

Krishna, V. *Auction Theory*. Academic Press, 2009.

Lengwiler, Y. “The Multiple Unit Auction with Variable Supply.” *Economic Theory*, Vol. 14 (1999), pp. 373–392.

LiCalzi, M. and Pavan, A. “Tilting the supply schedule to enhance competition in uniform-price auctions.” *European Economic Review*, Vol. 49 (2005), pp. 227–250.

Lyk-Jensen, S.V. and Chanel, O. “Retailers and Consumers in Sequential Auctions of Collectibles.” *Canadian Journal of Economics*, Vol. 40 (2007), pp. 278–95.

McAdams, D. “Adjustable supply in uniform price auctions: Non-commitment as a strategic tool.” *Economics Letters*, Vol. 95 (2007), pp. 48–53.

McAfee, R.P. and Vincent, D. “The Declining Price Anomaly.” *Journal of Economic Theory*, Vol. 60 (1993), pp. 191–212.

Mezzetti, C. “Sequential Auctions with Informational Externalities and Aversion to Price Risk: Decreasing and Increasing Price Sequences.” *The Economic Journal*, Vol. 121 (2011), pp. 990–1016.

Milgrom, P. and Weber, R. “A Theory of Auctions and Competitive Bidding.” *Econometrica*, Vol. 50 (1982), pp. 463–83.

—. “A Theory of Auctions and Competitive Bidding II.” In P. Klemperer (Ed.), *The Economic Theory of Auctions*. Edward Elgar Publishing (1999) .

Pesando, J.E. and Shum, P.M. “The auction market for modern prints: Confirmations, contradictions, and new puzzles.” *Economic Inquiry*, Vol. 46 (2008), pp. 149–159.

Said, M. “Sequential auctions with randomly arriving buyers.” *Games and Economic Behavior*, Vol. 73 (2011), pp. 236–243.

—. "Auctions with dynamic populations: Efficiency and revenue maximization." *Journal of Economic Theory*, Vol. 147 (2012), pp. 2419–2438.

Sørensen, S.T. "Sequential Auctions for Stochastically Equivalent Complementary Objects." *Economics Letters*, Vol. 91 (2006), pp. 337–342.

Weber, R. "Multiple-Object Auctions." In R. Engelbrecht-Wiggans, R. Stark, and M. Shubik (Eds.), *Competitive Bidding, Auctions, and Procurement*. New York University Press (1983) .