# Notes on Calculus

According to author Dave Barry

> Newton invented calculus, which is defined as "the branch of mathematics that is so scary it causes everybody to stop studying mathematics."[1]

Actually, calculus is a precise and efficient language that makes it easier to express some kinds of ideas. In microeconomics, calculus gives us a sharper understanding of (among other things) elasticity and pricing. You can learn microeconomics without calculus, but the logic is both fuzzier and more complex. The same is true in many other fields related to business: finance, statistics, operations. In finance, for example, calculus is the natural tool for understanding portfolio choice (the minimum variance portfolio) and bond duration (the sensitivity of price to yield). What follows is a short, (relatively) non-technical review of those aspects of calculus you'll need in this course.

■ **Executive summary.** Suppose $y = f(x)$. We find the maximum value of $y$ by setting the derivative of $f$ equal to zero: $f'(x) = 0$ (solve this for $x$). Our first application: $x$ is quantity produced and $y$ is profit; find the quantity that produces the greatest profit.

## Functions

In economics (and other fields, too), we often use relations between two variables: demand depends on price, cost depends on quantity produced, and so on. We call these relations "functions." More formally, a function $f$ assigns a (single) value $y$ to each possible value of $x$. We write it this way: $y = f(x)$. In a spreadsheet program, you might imagine setting up a table with a grid of values for $x$. The function would then be a formula that computes $y$ for each value of $x$.

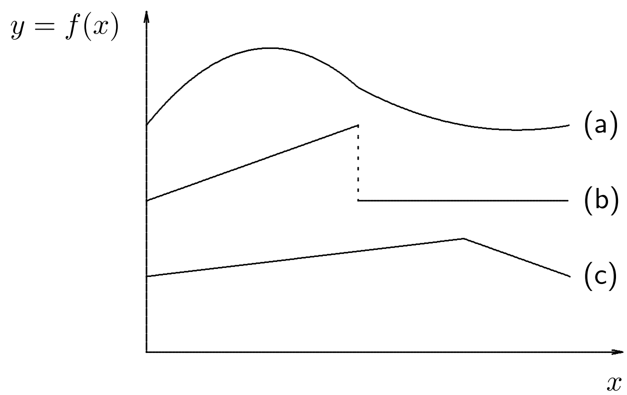
Perhaps the easiest way to think about a function is to draw it: put $x$ on the horizontal axis and plot the values of $y$ associated with each $x$ on the vertical axis. Some examples are given in Figure 1. We will generally be interested in functions that are "continuous" (they don't have "jumps," as in (b)) and "smooth" (they don't have "kinks," as in (c)).

## Slopes and derivatives

The "slope" of a function is a measure of how steep it is: the ratio of the change in $y$ to the change in $x$. For a straight line, we can find the slope by choosing two points and

1. "Great Moments in Science," *The Miami Herald*, March 16, 1997.

**Figure 1**
Three examples of functions




computing the ratio of the change in $y$ to the change in $x$. For most functions, though, the slope (meaning the slope of a straight line tangent to the function) is different at every point. Take function (a) for example. The slope is initially positive as the function increases, turns negative as the function slopes down, then turns positive again at the end. We could find the slope by drawing a tangent line at each point and laboriously computing their slopes, but there is an easier way.

The *derivative* $f'(x)$ of a function $f(x)$ gives us its slope at each point $x$ if the function is continuous and smooth. Formally, we say that the derivative is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

for $\Delta x$ close to zero, that is, for a "really small" change in $x$. (You can imagine doing this on a calculator or computer using a particular small number, and if the number is small enough your answer will be pretty close.) We express the derivative as $f'(x)$ or $\frac{dy}{dx}$. In the latter case, the $d$'s are intended to be suggestive of small changes, analogous to $\frac{\Delta y}{\Delta x}$ but with the understanding that we are talking about very small changes in $x$.

■ **Technical issues.** If a function has jumps or kinks, the notion of slope simply doesn't make sense at those points. Take function (b) in Figure 1. What is the slope where the function jumps (see the dotted line)? We could find the slope just to the left of the jump, or just to the right, but not at the point where it jumps. Similarly, it's not clear what the slope is at the kink in function (c). We say, in these cases, that the derivative doesn't exist. We will make sure that most of our functions are continuous and smooth so that this hardly ever happens.

So the derivative is a function $f'(x)$ that gives us the slope of a function $f(x)$ at every possible value of $x$. What makes this useful is that there are some relatively simple mechanical rules for finding $f'$ for some common functions $f$. Table 1 summarizes these rules for some common functions. If they're new to you, do not try to make sense of them. Take them as facts to be memorized and put to work.

■ **Examples.** Find the derivatives of the following functions [answers in brackets]:

**Table 1**
Rules for Computing Derivatives

| Rules for combinations of functions | | |
|---|---|---|
| Function $f(x)$ | Derivative $f'(x)$ | Comments |
| $g(x) + h(x)$ | $g'(x) + h'(x)$ | |
| $a\,g(x) + b\,h(x)$ | $a\,g'(x) + b\,h'(x)$ | $a$ and $b$ are numbers (constants) |
| $g(x)\,h(x)$ | $g(x)\,h'(x) + g'(x)\,h(x)$ | |
| $\frac{g(x)}{h(x)}$ | $\frac{g'(x)\,h(x) - g(x)\,h'(x)}{h(x)^2}$ | $h(x) \neq 0$ |
| $g\left(h(x)\right)$ | $g'\left(h(x)\right)\,h'(x)$ | "chain rule" |
| Rules for specific functions | | |
| Function $f(x)$ | Derivative $f'(x)$ | Comments |
| $a$ | 0 | $a$ is a number |
| $a\,x + b$ | $a$ | $a$ and $b$ are numbers |
| $a\,x^n$ | $n\,a\,x^{n-1}$ | $a$ is a number, $n$ an integer |
| $a\,e^{b\,x}$ | $b\,a\,e^{b\,x}$ | $e^x$ is the exponential function |
| $a\,\ln x$ | $\frac{a}{x}$ | $a$ is a number, $\ln$ means "natural log" |

- $2\,x + 27$ $[2]$
- $2\,x^2 + 3\,x + 27$ $[4\,x + 3]$
- $2\,x^2 + 3\,x - 14$ $[4\,x + 3]$
- $(x - 2)(2\,x + 7)$ $[4x + 3]$
- $\log(2\,x^2 + 3\,x - 14)$ $\left[\frac{4\,x+3}{2\,x^2+3\,x-14}\right]$
- $3\,x^8 + 13$ $[24\,x^7]$

## Finding the maximum of a function

Now to the bottom line. We will often find the value x that produces the maximum value of a function $f(x)$ for $x$ between (two numbers) $a$ and $b$. We do this by setting the derivative $f(x)$ equal to zero and solving for $x$. Why does this work? You can see in function (a) in Figure 1 that a function is flat (has zero slope) at a maximum. We simply put this insight to work. As usual, a little practice is worth more than words:

■ **Examples.** Find the maximum of each of these functions for $x \geq 0$ [answers in brackets: first the equation for the maximum, then the solution]:

- $x^2 - 2\,x$ $[2\,x - 2 = 0, x = 1]$
- $2\,\log x - x$ $[\frac{2}{x} - 1 = 0, x = 2]$

- $5x^2 - 2x + 11$ [$10x - 2 = 0, x = \frac{1}{5}$]
- function (a) in Figure 1 above [there are two points with zero slope, but only the first is a maximum]
- function (c) in the same figure [there's a maximum, but it's at a kink so we can't find it by setting the derivative equal to zero]

■ **Technical issues.** Does this always work? If we set the derivative equal to zero, do we always get a maximum? The answer, in a word, is no. Here are some of the things that could go wrong:

- The point might be a minimum, rather than a maximum. For example, in function (a) of Figure 1 the function has both a maximum and a minimum. Both have derivatives/slopes of zero.
- The maximum could be at one of the endpoints, $a$ or $b$. There's no way to tell without comparing your answer to $f(a)$ and $f(b)$.
- There may be more than one "local maximum."
- The slope might be zero without being either a maximum or a minimum: for example, the function might increase for a while, flatten out (with slope of zero), then start increasing again. An example is the function $f(x) = x^3$ at the point $x = 0$. (Draw it for yourself to make sure you understand the point.)

All of these things can happen in principle, but our job is to make sure they do not happen in this class. And they will not. (If you want to be extra careful, there are ways to check for each of these problems. One is the so-called second-order condition referred to in our notes on pricing.)

## Maximizing profit

Here's a concrete example of a common application in economics. Suppose a firm faces a demand for its product of $q = 10 - 2p$ ($q$ and $p$ being quantity and price, respectively). The cost of production is 2 per unit. What is the firm's profit? What level of output produces the greatest profit?

Answer. Profit is revenue ($p \times q$) minus cost ($2q$). The trick (and this isn't calculus) is to express it in terms of quantity. We use the demand curve to eliminate price from the expression for revenue: $p = \frac{10-q}{2}$ so $pq = \frac{10-q}{2}\, q$. Profit (expressed as a function of $q$) is therefore

$$\pi(q) = \frac{10-q}{2}\, q = 5q - \frac{1}{2}\, q^2 - 2q.$$

(Economists frequently use the Greek letter $\pi$ to represent the profit function.) To find the quantity associated with maximum profit, we set the derivative equal to zero:

$$\pi'(q) = 5 - q - 2 = 0,$$

so $q = 3$. What's the price? Look at the demand curve: if $q = 3$, then $p$ satisfies $3 = 10 - 2p$, so $p = \frac{7}{2}$.

Make sure you understand this example: problems like this will come up again.

## Further Reading

There are lots of good calculus books. Three good summaries are:

- Daniel Kleppner and Norman Ramsey, *Quick Calculus: A Self-Teaching Guide* (Second Edition), Wiley, 1985.
- Bernard Zandy and Jonathan White, *Cliffs Quick Review Calculus*, Wiley, 2001.
- Silvanus Thompson and Martin Gardner, *Calculus Made Easy,* St Martins Press, 1998. Our personal favorite.