18 | Combinatorial-Form Games

Game theory, the formal analysis of strategic interaction between players (firms, armies, countries, etc.) can be divided into two branches: combinatorial-form games and non-cooperative games. Non-cooperative games are the most common entry point into game theory. In a non-cooperative game, the sequence of moves by each player, as well as the choices available to them, are well-determined. Having decided what the game is, the goal is then to predict the strategies that one would expect each player to choose.

Non-cooperative games are most appropriate when the situation at hand is characterized by clear rules determining what players can choose. Consider, for example, the auction of offshore oil drilling rights in a particular area. It is well established that each player can (a) invest a certain amount to determine how good the drilling prospects are; (b) submit a sealed bid by a certain date. The winner is then selected according to some pre-specified rules (normally the highest bid).

Combinatorial-form games, by contrast, are best suited to study situations when there is no clear sequence of moves, that is, when the game being played has no clear "rules." Consider the different negotiations involved in producing and selling a television show. Actors negotiate with producers, producers negotiate with networks — or perhaps the three negotiate simultaneously. There is no pre-set protocol for how these negotiations take place. It would therefore be difficult — and somewhat arbitrary — to model these as a non-cooperative game, with a precise, pre-specified, sequence of moves. Combinatorial-form games allow us to side-step this problem, as long as we have an idea of the value that various combinations of players would be able to achieve. Once we have that, the goal is then to predict the value that is captured by each player.

In this note I introduce the analysis of combinatorial-form games. Throughout, I illustrate the various concepts by solving a specific example motivated by negotiations over payments between a producer, a network and the actors involved in producing an extra season of a well-known television show. The companion case, *TV Power Games: Friends and Law & Order*, provides two real-world examples of such negotiations.

Forthcoming in *The Economics of Entertainment and Sports: Concepts and Cases.*

The value function

In most real-world situations, there are a large number of players interacting with each other in one way or another. The making of a TV show involves actors, writers, camera operators, directors, and so forth. To keep the analysis tractable, I will focus on three central players: the actors, who I assume act as one player, denoted by A; the producer, denoted by P; and the network, denoted by B (B for buyer, since, to be consistent with standard notation, I reserve N for the set of all players).

The first thing we need to do is to determine the value attained by each possible subset of players. This is known as the value function, or the characteristic function, and denoted by $v(S)$, where S is a set of players (for example, $S = \{A, P\}$ is the set comprised of actor and producer).

Let us first consider the value attainable by all players when assembled together, $N = \{A, P, B\}$. If a deal goes through, that is, if cast, producer and network agree on running the show for an extra season, then the following value is generated on a per-episode basis:

- Advertising revenue during first-runs: $6 million (6 minutes at $1 million per minute).
- Indirect network revenue (e.g., promos for other network shows): $3 million.
- Reruns: $4 million.

This leads to a total of $13 million per episode. We note that the revenue from advertising on first-runs is received by the network, whereas the revenue from reruns belongs to the producer. I will return to this later.

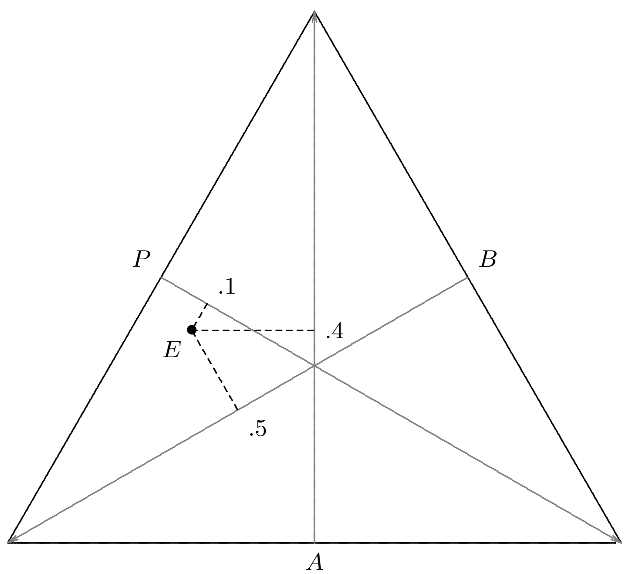
Next we need to determine what each player's best alternative is if a deal does not go through (also known as best alternative to a negotiated agreement, or BATNA).

- The network can use the time slot to run a show that will bring in net revenues to the tune of $1.5 million.
- The producer can work on a different project, valued at $3 million (net profits).
- The actors have a series of alternative projects that are worth to them a total of $1 million.

Finally, we need to determine what value players can generate if only two of them are on board. Very frequently, the show is effectively jointly owned by the producer and the network. This implies that, if either of them is not on board, then the show is discontinued. It follows that the value obtained by each player corresponds to the value of their outside option (going alone); and the value obtained by any two players is simply the sum of their outside option values.

We are left with the possibility of the producer and the network making a deal without the actors — that is, using a separate cast. Let α be the percent loss in value from losing the current cast. It follows that running the show with a different set of actors generates value $(1 - \alpha) \times \$13$ million. This is only true if α is small, that is, if the replacement cast is almost as good as the current one. If α is very high, however, then producer and network are better off by dropping the show altogether and moving on to their best alternative projects. Specifically, given the values above, the producer and the network prefer to drop the show when the current cast is not on board and α is greater that $8.5/13 \approx 65\%$. For the time being, assume that $\alpha = 20\%$.

Figure 18.1
The two-dimentional simplex (three player game).



Pulling all of these numbers together, we have

$$
\begin{aligned}
v(\{A,P,B\}) &= 13 \\
v(\{A\}) &= 1 \\
v(\{P\}) &= 3 \\
v(\{B\}) &= 1.5 \\
v(\{A,P\}) &= v(\{A\}) + v(\{P\}) = 4 \\
v(\{A,B\}) &= v(\{A\}) + v(\{B\}) = 2.5 \\
v(\{P,B\}) &= (1-\alpha)\, v(\{A,P,B\}) = 10.4
\end{aligned}
\tag{18.1}
$$

The question that combinatorial game theory addresses is: starting from these values, what distribution of value among the various players would we expect? There are several possible answers to this question. I will look at the two most common approaches. Before doing so, I introduce a graphical device to represent the distribution of value among players.

Suppose that a deal goes through, that is, the show is to be run. A value of \$13 needs to be distributed among players. We say that the division is efficient if the values taken by each player add up to \$13, that is, if no value is lost in the process of distributing it among the players. Letting x_A, x_P, x_B be the payoff received by each player, we have

$$x_A + x_P + x_B = 13$$

An expedient way of representing the values x_i is to draw a simplex, which in the case of three players corresponds to a triangle. Figure 18.1 shows the simplex for the game at hand. It has three axes, one for each player. The payoff for player A (actors), for example, is measured by the distance of each point with respect to the A axis, the same for players P (producer) and B (buyer). For example, suppose that the split of the total value is as follows: 10% for P, 50% for B and 40% for A. This is represented by point E in Figure 18.1.

We next deal with two game-theory concepts that determine the values achieved by each player for a given value functions $v(S)$.

The Core

The core is a set of value splits (x_A, x_P, x_B) that satisfies the following conditions. First, it is efficient, that is, the sum of all players' payoffs is equal to the value at stake. Second, for every set of players it must be that the sum of their payoffs is at least as large as the value that they could attain on their own. Formally,

$$\begin{aligned} &\sum_{i \in C} x_i = v(N) \\ &\sum_{i \in S} x_i \geq v(S) \qquad \forall\, S \subset C \end{aligned} \tag{18.2}$$

where N is the set of all players: in the present example, $N = \{A, P, B\}$.

Although in general the above corresponds to a large set of inequalities ($n!$, where n is the set of players), in practice the crucial inequalities are of two types:

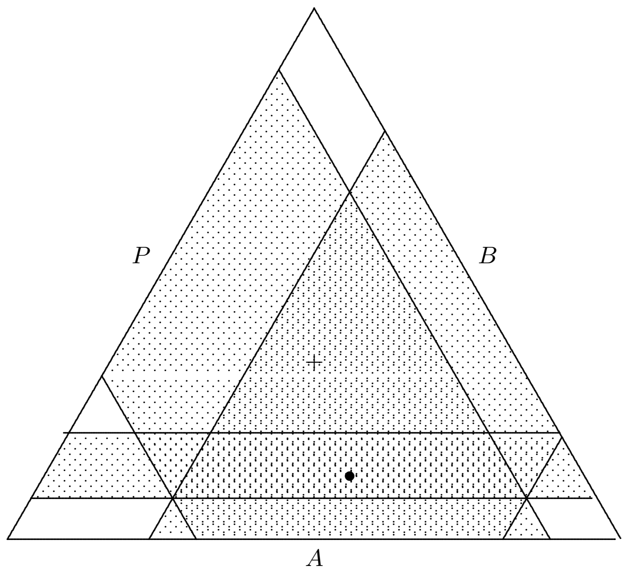
$$\begin{aligned} x_i &\geq v(\{i\}) \\ x_i &\leq v(N) - v(N \backslash \{i\}) \end{aligned} \tag{18.3}$$

Both of these inequalities can be derived from (18.2). The first set of inequalities in (18.3) corresponds to the *individual rationality* or *outside option* constraints: each player must get at least as much as he or she would obtain by going solo, that is, each player must get at least his or her BATNA. The second set of constraints corresponds to the principle of *marginal contribution*: each player can only get as much as he or she contributes to the deal, a value also known as *added value*. I next show how it can derived from (18.2). For simplicity, I focus on the payoff for player A. From the first equation in (18.2), we know that $x_A + x_P + x_B = v(N)$. We thus have that $x_A = v(N) - x_P - x_B$. But one of the inequalities in (18.2) corresponds to $x_P + x_B \geq v(\{P, B\})$. This finally implies that $x_A \leq v(N) - v(\{P, B\})$.

Going beyond the math, what the principle of marginal contribution states is that player A cannot get more than his or her marginal contribution to the ensemble $\{A, P, B\}$. If player A were getting more than his or her marginal contribution, that is, if $x_A > v(N) - v(\{P, B\})$, then players P and B could simply walk away and earn more on their own. That is, $x_A > v(N) - v(\{P, B\})$ implies that $x_P + x_B < v(\{P, B\})$; and the latter inequality implies that the proposed deal is not stable.

Figure 18.2 illustrates the core of our three-player game. There are three shaded "bands," one for each player, that correspond to the set of conditions (18.3), that is, the conditions that a player earn more than his or her outside value but less than his or her marginal contribution. The intersection of these three bands, the trapezoids in darker shade, corresponds to the core of the game. All points in this region satisfy the conditions that no player or set of players would wish to renege on the agreement.

Figure 18.2
Equilibrium values. The Core corresponds to the region shaded darker. The Shapley value corresponds to •, whereas a + indicates the point of equal division of value.



The Shapley value

The core is a simple and intuitive notion with which to analyze games. However, one limitation is that it indicates a *set* of pie splitting possibilities, rather than a single value. (Sometimes, the core indicates a single value, but then again, sometimes it is an empty set.) The *Shapley value*, by contrast, indicates a *single* solution in every situation.

Before going into the general mathematical formula for the Shapley value, it helps to explain its intuition by considering the specific example introduced above. Suppose that players enter the deal in a certain order and that each player is paid according to his or her contribution to the group of players already assembled at the time the new player joins in (also knows as the player's *added value*). For example, suppose the order is given by A, then P, then B. In this case, Player A adds $v(\{A\}) - v(\emptyset) = v(\{A\}) = 1$. Player P's incremental contribution is given by $v(\{A, P\}) - v(\{A\}) = 4 - 1 = 3$. Finally, Player B's incremental contribution is given by $v(\{A, P, B\}) - v(\{A, P\}) = 13 - 4 = 9$.

Now let us do the same calculation for every possible sequence of players, that is, for every permutation of A, P and B. The following table shows the values obtained by each player in each case. Recall that each player is paid what his incremental contribution is for each particular sequence of players. For example, in the PBA sequence, player A receives the difference between $v(\{A, P, B\})$, the value when all players are together, and $v(\{P, B\})$, the value when player A is absent from the previous set. In this particular case, player A gets $13 - 10.4 = 2.6$.

Notice that, when we determine the value obtainable by a given set of players, $v(S)$, the order of the elements in S does not matter. At this stage, all we want to know is the size of the pie. However, in the rows of the next table, the order of players is crucial, for it is this order that determines the split of the pie.

Since there are three players, there are $3! = 6$ permutations, that is, 6 possible sequences of players. The table below gives each player's payoff for each permutation, assuming that each player is paid according to his or her incremental contribution.

	Incremental value		
Sequence of players	A	P	B
APB	1	3	9
ABP	1	10.5	1.5
PAB	1	3	9
PBA	2.6	3	7.4
BAP	1	10.5	1.5
BPA	2.6	8.9	1.5
Average	1.53	6.48	4.98

Different permutations lead to different payoffs assigned to each player. (Notice in particular that a player is typically strictly better off the later he comes in the sequence — because then his incremental contribution is greater.) Which of the six permutations is more reasonable? One way to think about the concept of Shapley value is to say that all sequences are equally likely. As a result, we should assign to each player the average of his or her payoff across all possible permutations. This is given by the last row in the above table. For example, the Shapley value of player A is 1.53.

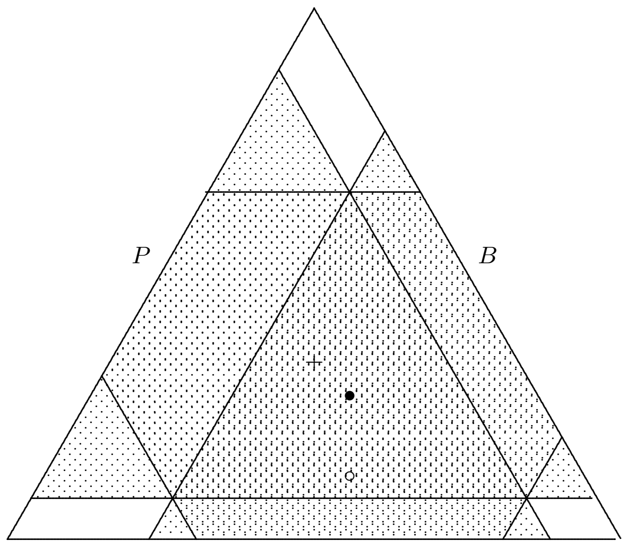
More generally, player i's Shapley value is given by

$$x_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|\,!\,(n - |S| - 1)\,!}{n\,!} \; \big(v(S \cup \{i\}) - v(S)\big) \tag{18.4}$$

where $|S|$ is the cardinal of S (that is, the number of elements in S), $n!$ is the factorial of n (that is, $n \times (n-1) \times \ldots \times 2 \times 1$), and n is the number of players. This is a mathematics mouthful, so let us take it piece by piece. Let us start with the right-most term, $v(S \cup \{i\}) - v(S)$. This measures the player i's incremental contribution (added value) when it joins an existing set S of players. The symbol $\sum_{S \subseteq N \setminus \{i\}}$ means that we want to sum those incremental contributions for all possible sets S, that is, for all possible "positions" by player i in the sequence of players. Specifically, $N \setminus \{i\}$ means the set of all players except player i; $S \subseteq N \setminus \{i\}$ means that S is a subset of $N \setminus \{i\}$; and $\sum_{S \subseteq N \setminus \{i\}}$ means the sum of all instances when S is a subset of $N \setminus \{i\}$. If $S = N \setminus \{i\}$, then we are in the case when player i is the last one to "move;" if, by contrast, S is a strict subset of $N \setminus \{i\}$, then player i "moves" earlier in the sequence of players.

Finally, the ugly expression after the summation mark, $\frac{|S|\,!\,(n-|S|-1)\,!}{n\,!}$, tells what weight to place on a certain set S. Take for example the computation of player A's value. If $S = \{P, B\}$, then player A is the last to move. How many sequences are there where player A is the last to move? Two: PBA and BPA. In fact, in this case $|S| = 2$ and $n! = 3 \times 2 \times 1 = 6$. It follows that $\frac{|S|\,!\,(n-|S|-1)\,!}{n\,!} = \frac{1}{3}$, which is exactly the weight we want to give to this case (two our of six possible permutations). More generally, (18.4) gives the average computed in the above table.

Figure 18.3
Equilibrium values when α increases from 20% to 80%. A $\circ$ denotes the previous Shapley value, whereas a $+$ denotes the equal-split solution.

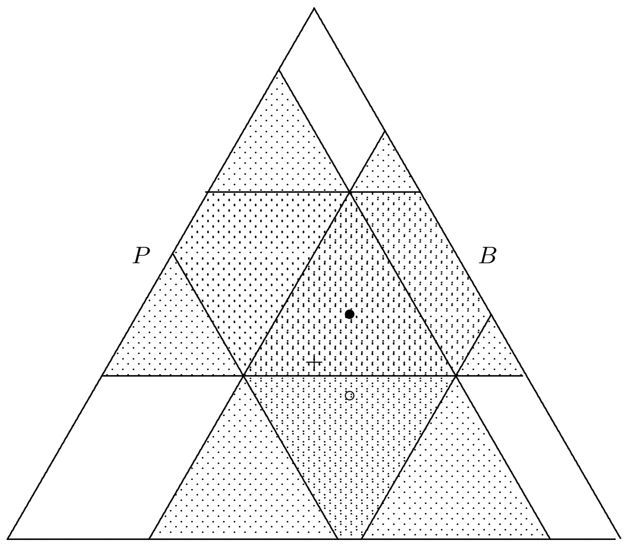


The Shapley values for each player in our ongoing example is plotted with a bullet in Figure 18.2. Notice that in this case the Shapley value lies inside the Core, that is, it is one of the values that satisfy the Core conditions. This is not very surprising, since the Shapley value and the Core are closely related: both are based on the notion of a player's incremental contribution (added value). Specifically, consider the case when S is the empty set. Then $v(S \cup \{i\}) - v(S) = v(\{i\})$, the value obtained by i alone. This is what we normally refer to as player i's outside option, or BATNA. If S represents all players other than player i, then $v(S \cup \{i\}) - v(S)$ tells us how much is lost when player i walks away from the complete set of players. This is nothing but the concept of marginal contribution (or value added), introduced earlier.

So, like the Core — see Equations (18.3) — the Shapley value takes into account a player's outside option and marginal contribution. Unlike the Core, which uses outside options and marginal contributions as bounds on a player's value, the Shapley value is determined as a weighted average of marginal contributions (not only marginal contributions to the whole set of players but also any subset of players). Like the core, the Shapley value is individually rational, that is, $x_i \geq v(\{i\})$; and efficient, that is, $\sum x_i = v(N)$.

Finally, the Shapley values of each player usually belong to the Core, but not necessarily so. In fact, as mentioned before, there may be games where no solution belongs to the Core, whereas the Shapley value always exists — and is unique, which helps in the process of determining the impact of changing conditions, to which I turn next.

Figure 18.4
Equilibrium values when $v(\{A\})$ increases from 1 to 4. A $\circ$ denotes the previous Shapley value, whereas a $+$ denotes the equal-split solution.



Changing conditions

Now that we have introduced the two central concepts from combinatorial-form game theory, let us examine how they can be used to analyze the impact of various market changes. Suppose that, as a show develops and audiences become better acquainted with the actors, the loss in value from replacing the cast increases from 20% to 80%. In term of the values in (18.1), we now have

$$v(\{P, B\}) = v(\{P\}) + v(\{B\}) = 4.5$$

(Notice that the formula $v(\{P, B\}) = (1 - \alpha)\, v(\{A, P, B\})$ no longer applies because such value would be so low that P and B would find it is better to scratch the show completely.) Otherwise, the values in (18.1) remain unchanged.

In terms of the Shapley value, the new equilibrium is given by

$$x_A = 3.5$$
$$x_P = 5.5$$
$$x_B = 4$$

This corresponds to a price per episode (paid by the network to the producer) of $5 million dollars per episode. Why? Because we know the network receives revenues of $9 million (advertising and indirect network revenue). In order for its value to be 4, it must pay a price of $9 - 4 = 5$.

Figure 18.3 shows how the Shapley value changes as α changes. An increase in α from .2 to .8, meaning that the cast becomes more irreplaceable, leads to an equilibrium shift from $\circ$ to $\bullet$. The shift is more or less perpendicular to the A axis, that is, it essentially consists of an increase in the actors' payoff that is compensated for by a decrease in producer's and buyer's payoff. Notice that the Core also changes, in particular it now includes a wider (and higher) range of A's payoff. Finally, notice that the Shapley value is still inside the Core.

Suppose now that, as the actors become better known, better outside opportunities arise. Frequently, these corresponds to feature movies which would prevent the actors from continuing their regular schedule with the show. As a result of this improvement in outside options, we now have $v(\{A\}) = 4$. Other than this and $v(\{P, B\}) = 4.5$, the values in (18.1) remain unchanged.

In terms of the Shapley value, the new equilibrium is given by

$$x_A = 5.5$$
$$x_P = 4.5$$
$$x_B = 3$$

This corresponds to a price per episode (paid by the broadcaster to the producer) of \$6 million dollars per episode. Figure 18.4 shows the effect of the increase in A's BATNA on the Core and Shapley value. As before, the Shapley value point moves upwards, indicating a higher equilibrium payoff for A. Notice that the new equilibrium lies to the northeast of $+$, the equal-split point. This shows that, in the new equilibrium, the buyers receives less than an equal share. One can also see that P (the producer) receives about the same as in an equal-split solution, whereas the actors (A) receive more than in the equal-split solution.

More generally, the above examples show two factors that influence the extent to which a player (in this case the actor or actors) is able to capture value: first, the degree to which the player is irreplaceable, specifically as given by $v(N) - v(N\backslash\{i\})$ (which in our example is measured by α); and second, the value of their outside option, which is given by $v(\{i\})$.